MIFP-04-20 UPR-1096-T
hep-th/0411045
November 2004

Charged Rotating Black Holes in Four-Dimensional Gauged and Ungauged Supergravities

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Abstract

We study four-dimensional non-extremal charged rotating black holes in ungauged and gauged supergravity. In the ungauged case, we obtain rotating black holes with four independent charges, as solutions of $\mathcal{N}=2$ supergravity coupled to three abelian vector multiplets. This is done by reducing the theory along the time direction to three dimensions, where it has an $O(4,4)$ global symmetry. Applied to the reduction of the uncharged Kerr metric, $O(1,1)^4 \subset O(4,4)$ transformations generate new solutions that correspond, after lifting back to four dimensions, to the introduction of four independent electromagnetic charges. In the case where these charges are set pairwise equal, we then generalise the four-dimensional rotating black holes to solutions of gauged $\mathcal{N}=4$ supergravity, with mass, angular momentum and two independent electromagnetic charges. The dilaton and axion fields are non-constant. We also find generalisations of the gauged and ungauged solutions to include the NUT parameter, and for the ungauged solutions, the acceleration parameter too. The solutions in gauged supergravity provide new gravitational backgrounds for a further study of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence at non-zero temperature.

[1] Research supported in part by DOE grant DE-FG03-95ER40917.

[2] Research supported in part by DOE grant DE-FG02-95ER40893, NSF grant INTO3-24081, and the Fay R. and Eugene L. Langberg Chair.

Contents

1 Introduction

Rotating charged black hole solutions of ungauged supergravity play an important role in the microscopic study of black hole entropy. It turns out that the microscopic properties can be addressed quantitatively not only for BPS black holes, but also for black holes that are close to extremality. (For a recent review see [1], and references therein.) The prerequisite for these studies is to obtain explicit black hole solutions on the supergravity side. These solutions are typically characterised by multiple electromagnetic charges, in addition to the mass and angular momenta. In five dimensions, such explicit non-extremal solutions, specified by their mass, two angular momentum parameters and three charges, were found in [2]. They were obtained by employing a solution-generating technique using global symmetries of a four-dimensional theory that was obtained by reducing from five dimensions along the time direction. The first examples of electrically-charged rotating black holes in ungauged supergravity theories were obtained, by using such a solution-generating technique, in [3].

Employing analogous methods, the explicit metric and scalar fields for four-dimensional four-charge rotating black holes were obtained in [4]. Unfortunately, the explicit form of the four gauge potentials for this solution was not given explicitly in [4]. Further work, again using the same solution-generating procedure, produced black holes in $4 \le D \le 9$ dimensions parameterised by their mass, two charges, and their $[(D-1)/2]$ angular momentum parameters [5]. These solutions of ungauged supergravity all provide gravitational backgrounds for the microscopic study of black hole entropy within the string theory framework.

By contrast, black holes in gauged supergravity provide gravitational backgrounds that are relevant to the AdS/CFT correspondence. In particular, such non-extremal solutions play an important role in the study of the dual field theory at non-zero temperature. (An early study of the implications of static charged AdS black holes [6] in the dual theory was carried out in [7, 8]. For recent related work see [9, 10, 11] and references therein.) However, the explicit form of charged AdS black hole solutions that are also rotating has remained elusive until recently. In [12, 13] the first examples of non-extremal rotating charged AdS black holes in five-dimensional $\mathcal{N} = 4$ gauged supergravity were obtained, in the special case where the two angular momenta J_i are set equal. These solutions are characterised by their mass, three electromagnetic charges, and the angular momentum parameter $J = J_1 = J_2$. By taking appropriate limits, one obtains the various supersymmetric charged rotating $D = 5$ black holes obtained in [14, 15, 16]. If instead the charges are set to zero, the solutions reduce to the rotating AdS_5 black hole constructed in [17], with $J_1 = J_2$.

In four dimensions, there should exist rotating black hole solutions in gauged $\mathcal{N} = 8$ supergravity with four independent electromagnetic charges. Until now, the only known solutions of this type were the Kerr-Newman-AdS black holes [18, 19], which correspond to setting the four electromagnetic charges equal.

One of the purposes of this paper is first to construct the complete and explicit form of the general rotating black holes of four-dimensional *ungauged* supergravity, with four independent electromagnetic charges. They can be viewed as solutions in ungauged $\mathcal{N} = 2$ supergravity coupled to three vector multiplets, which in turn can be embedded in $\mathcal{N} = 8$ maximal supergravity. We employ a solution-generating technique in which the $\mathcal{N} = 2$ theory is reduced to three dimensions on the time direction, where it has an $O(4,4)$ global symmetry. By acting on the reduction of the uncharged Kerr solution with an $O(1,1)^4 \subset O(4,4)$ subgroup of the global symmetry, we obtain a new solution that lifts back to a solution of the four-dimensional theory with four independent electromagnetic charges. In this formulation, two of the $U(1)$ charges are electric and two are magnetic. By obtaining the explicit form of the four $U(1)$ gauge potentials, as well as the other fields, we therefore complete the results in [4], where the metric and scalar fields were found. We then apply the same generating technique to generalise these results by the inclusion also of the NUT parameter, and the acceleration parameter.

The second goal of this paper is to obtain charged rotating black hole solutions in four-dimensional *gauged* supergravity. We have been able to do this in the case where the four charges of the ungauged theory are first set pairwise equal. With this restriction, we are able to conjecture, and then explicitly verify, the expression for the two-charge rotating black hole solutions of $\mathcal{N} = 4$ gauged supergravity. The solutions have varying dilaton and axion fields.

The paper is organised as follows. In Section 2 we describe the solution-generating technique for constructing charged rotating black holes of four-dimensional ungauged supergravity. In fact the procedure can be used to introduce four independent charges in any Ricci-flat four dimensional metric admitting a timelike Killing vector. In Section 3 we present the explicit form of the four-charge rotating black hole solution, generated from the Kerr metric, and give its specialisation to the case where the charges are set pairwise equal. In Section 4 we present the generalisation of this latter case to a solution in $\mathcal{N} = 4$ gauged supergravity. In Section 5, generalisations of both the gauged and ungauged supergravity solutions are found, in which in addition the NUT parameter is non-zero. In Section 6 we obtain a further generalisation of the ungauged supergravity solutions to include also the

acceleration parameter. In Appendix A we give a matrix realisation of the generators of $O(4,4)$, which is helpful for implementing the explicit transformations in three dimensions. In Appendix B we give the explicit form of the rotating four-charge solution with the NUT parameter for the ungauged case. Appendix C contains a discussion of the supersymmetry of extremal rotating AdS black holes in four dimensions. Concluding remarks are given in Section 7.

2 Charge-Generating Procedure

In this section, we set up the basic formalism for generating four-dimensional configurations carrying 4 independent charges, that are solutions of ungauged $\mathcal{N}=2$ supergravity coupled to three vector multiplets. This theory, and hence also its solutions, can be consistently embedded in four-dimensional $\mathcal{N}=8$ supergravity. The procedure involves starting with an uncharged four-dimensional solution that has a timelike Killing vector $\partial/\partial t$, and reducing it to three dimensions on the t direction. The reduction of the $\mathcal{N}=2$ theory itself yields a three-dimensional theory with an $O(4,4)$ global symmetry, after all the three-dimensional vector fields have been dualised to axions. By acting with an $O(1,1)^4$ subgroup of $O(4,4)$ on the dimensionally reduced solution, we generate new solutions involving four parameters δ_i characterising the $O(1,1)^4$ transformation. Upon undualising the transformed dualised axions back to vectors again, and lifting back to $D=4$, we thereby arrive at supergravity solutions carrying 4 electromagnetic charges, parameterised by the δ_i.

In this section, we shall present the three-dimensional results for the reduction and $O(1,1)^4$ transformation of a general four-dimensional uncharged solution with a timelike Killing vector $\partial/\partial t$. One cannot abstractly "undualise" the three-dimensional scalars that originate from vectors in $D=4$ (or from the Kaluza-Klein vector), since dualisation is intrinsically a non-local procedure. However, once one has an explicit solution, the process of undualisation can be implemented explicitly. Thus, in subsequent sections we shall apply these results to particular cases, and implement the complete and explicit construction of the charged four-dimensional solutions. (A solution-generating technique of the type we are using here was first employed in [3], to obtain electrically charged rotating black holes in ungauged supergravity.) As we shall see later, our solutions will carry two electric and two magnetic charges.

2.1 $O(4,4)$ symmetry of the reduced $D=3$ theory

The four-dimensional Lagrangian for the bosonic sector of the $\mathcal{N}=2$ supergravity coupled to three vector multiplets can be written as[1]

$$\begin{aligned}
\mathcal{L}_4 &= R\,{*1\!\!1} - \tfrac12 {*d\varphi_i}\wedge d\varphi_i - \tfrac12 e^{2\varphi_i}\,{*d\chi_i}\wedge d\chi_i - \tfrac12 e^{-\varphi_1}\Big(e^{\varphi_2-\varphi_3}\,{*\hat F_{(2)1}}\wedge \hat F_{(2)1} \\
&\quad + e^{\varphi_2+\varphi_3}\,{*\hat F_{(2)2}}\wedge \hat F_{(2)2} + e^{-\varphi_2+\varphi_3}\,{*\hat{\mathcal{F}}^1_{(2)}}\wedge \hat{\mathcal{F}}^1_{(2)} + e^{-\varphi_2-\varphi_3}\,{*\hat{\mathcal{F}}^2_{(2)}}\wedge \hat{\mathcal{F}}^2_{(2)}\Big) \\
&\quad -\chi_1\,(\hat F_{(2)1}\wedge \hat{\mathcal{F}}^1_{(2)} + \hat F_{(2)2}\wedge \hat{\mathcal{F}}^2_{(2)})\,,
\end{aligned} \tag{1}$$

where the index i labelling the dilatons φ_i and axions χ_i ranges over $1\le i\le 3$. The four field strengths can be written in terms of potentials as

$$\begin{aligned}
\hat F_{(2)1} &= d\hat A_{(1)1} - \chi_2\, d\hat{\mathcal{A}}^2_{(1)}\,,\\
\hat F_{(2)2} &= d\hat A_{(1)2} + \chi_2\, d\hat{\mathcal{A}}^1_{(1)} - \chi_3\, d\hat A_{(1)1} + \chi_2\,\chi_3\, d\hat{\mathcal{A}}^2_{(1)}\,,\\
\hat{\mathcal{F}}^1_{(2)} &= d\hat{\mathcal{A}}^1_{(1)} + \chi_3\, d\hat{\mathcal{A}}^2_{(1)}\,,\\
\hat{\mathcal{F}}^2_{(2)} &= d\hat{\mathcal{A}}^2_{(1)}\,.
\end{aligned} \tag{2}$$

Note that we are placing hats on the four-dimensional field strength and gauge potentials, to distinguish them from the three-dimensional fields.

The four-dimensional theory can be obtained from six-dimensions, by reducing the bosonic string action

$$\mathcal{L}_6 = R\,{*1\!\!1} - \tfrac12 e^{-\sqrt2\phi}\,{*F_{(3)}}\wedge F_{(3)} \tag{3}$$

on T^2. Thus the four-dimensional Lagrangian itself has an $O(2,2)\sim SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ global symmetry, which enlarges at the level of the equations of motion to include a third $SL(2,\mathbb{R})$ factor when electric/magnetic S-duality transformations are included. We are going to reduce it one stage further, to $D=3$. If left in its raw form, the three-dimensional Lagrangian would have an $O(3,3)$ global symmetry. However, if the 1-form potentials in $D=3$ are dualised to axions (so that there are only dilatons and axions, plus the metric, in $D=3$), then as is well known, the global symmetry will be enhanced to $O(4,4)$. The reduction from $D=4$ to $D=3$ will be performed on the time coordinate. This will imply that the coset parameterised by the dilatons and axions will not be $O(4,4)/(O(4)\times O(4))$, as would be the case for a spacelike reduction, but instead $O(4,4)/O(4,\mathbb{C})$.

To proceed, we first reduce the fields in the Lagrangian (1), according to the standard Kaluza-Klein reduction scheme adapted to the case of a timelike reduction. Thus we write

[1]Our conventions for dualisation are that a p-form ω with components defined by $\omega = 1/p!\,\omega_{i_1\cdots i_p}\,dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ has dual $*\omega$ with components $(*\omega)_{i_i\cdots i_{D-p}} = 1/p!\,\epsilon_{i_1\cdots i_{D-p}j_i\cdots j_p}\,\omega^{j_1\cdots j_p}$.

the following reduction ansätze for the metric and for 1-form potentials:

$$
\begin{aligned}
d\hat{s}_4^2 &= -e^{\varphi_4}\,(dt+\mathcal{B}_{(1)})^2 + e^{-\varphi_4}\,ds_3^2\,, &(4)\\
\hat{A}_{(1)} &= A_{(1)} + A_{(0)}\,(dt+\mathcal{B}_{(1)})\,. &(5)
\end{aligned}
$$

The field strengths reduce according to the rule

$$
\hat{F}_{(2)} = F_2 + F_1 \wedge (dt+\mathcal{B}_{(1)})\,. \tag{6}
$$

In order to abbreviate the description, we shall directly present the fully-dualised form of the three-dimensional Lagrangian that results from reducing (1) according to this scheme, and then indicate afterwards how the three-dimensional fields are related to the four-dimensional ones. We find that the fully-dualised three-dimensional Lagrangian can be written as

$$
\begin{aligned}
e^{-1}\,\mathcal{L}_3 \;=\;& R - \tfrac12(\partial\varphi_i)^2 - \tfrac12 e^{2\varphi_1}\,(\partial\chi_1)^2 - \tfrac12 e^{2\varphi_2}\,(\partial\chi_2)^2 - \tfrac12 e^{3\varphi_1}\,(\partial\chi_3)^2\\
&-\tfrac12 e^{-2\varphi_4}\,(\partial\chi_4 + \sigma_1\,\partial\psi_1 + \sigma_2\,\partial\psi_2 + \sigma_3\,\partial\psi_3 + \sigma_4\,\partial\psi_4)^2\\
&+\tfrac12 e^{-\varphi_1+\varphi_2-\varphi_3-\varphi_4}\,(\partial\sigma_1 - \chi_2\,\partial\sigma_4)^2\\
&+\tfrac12 e^{-\varphi_1+\varphi_2+\varphi_3-\varphi_4}\,(\partial\sigma_2 + \chi_2\,\partial\sigma_3 - \chi_3\,\partial\sigma_1 + \chi_2\,\chi_3\,\partial\sigma_4)^2\\
&+\tfrac12 e^{-\varphi_1-\varphi_2+\varphi_3-\varphi_4}\,(\partial\sigma_3 + \chi_3\,\partial\sigma_4)^2 + \tfrac12 e^{-\varphi_1-\varphi_2-\varphi_3-\varphi_4}\,(\partial\sigma_4)^2\\
&+\tfrac12 e^{\varphi_1-\varphi_2+\varphi_3-\varphi_4}\,(\partial\psi_1 + \chi_3\,\partial\psi_2 - \chi_1\,\partial\sigma_3 - \chi_1\,\chi_3\,\partial\sigma_4)^2\\
&+\tfrac12 e^{\varphi_1-\varphi_2-\varphi_3-\varphi_4}\,(\partial\psi_2 - \chi_1\,\partial\sigma_4)^2\\
&+\tfrac12 e^{\varphi_1+\varphi_2-\varphi_3-\varphi_4}\,(\partial\psi_3 - \chi_2\,\partial\psi_2 - \chi_1\,\partial\sigma_1 + \chi_1\,\chi_2\,\partial\sigma_4)^2\\
&+\tfrac12 e^{\varphi_1+\varphi_2+\varphi_3-\varphi_4}\,(\partial\psi_4 + \chi_2\,\partial\psi_1 - \chi_3\,\partial\psi_3 - \chi_1\,\partial\sigma_2 + \chi_2\,\chi_3\,\partial\psi_2\\
&\qquad\qquad -\chi_1\,\chi_2\,\partial\sigma_3 + \chi_1\,\chi_3\,\partial\sigma_1 - \chi_1\,\chi_2\,\chi_3\,\partial\sigma_4)^2\,,
\end{aligned} \tag{7}
$$

where now the i index ranges over $1 \le i \le 4$. Note that the last eight terms have the non-standard sign for their kinetic terms, in consequence of the timelike reduction, and the subsequent dualisations in a Euclidean-signature metric. The axion χ_4 corresponds to the dual of the Kaluza-Klein vector $\mathcal{B}_{(1)}$ in (4); the axions σ_i correspond to the components $A_{(0)}$ (as in (5)) of the reductions of the 4 four-dimensional potentials, in the order $(A_{(1)1}, A_{(1)2}, \mathcal{A}^1_{(1)}, \mathcal{A}^2_{(1)})$; and the axions ψ_i correspond to the dualisations of the three-dimensional 1-forms $A_{(1)}$ in (5), taken in the same order as the σ_i.

In detail, the dualisations are performed as follows.[2] The field strength $G_{(2)} = d\mathcal{B}_{(1)}$ for

[2]These results are obtained by applying the standard procedure of introducing the dual potential as a Lagrange muliplier for the Bianchi identity of the original field strength.

the Kaluza-Klein 1-form is replaced by

$$e^{2\varphi_4} *G_{(2)} = d\chi_4 + \sigma_1\, d\psi_1 + \sigma_2\, d\psi_2 + \sigma_3\, d\psi_3 + \sigma_4\, d\psi_4\,. \tag{8}$$

The four field strengths coming from the four field strengths in four dimensions are replaced by

$$\begin{aligned}
-e^{-\varphi_1+\varphi_2-\varphi_3+\varphi_4} *F_{(2)1} &= d\psi_1 + \chi_3\, d\psi_2 - \chi_1\, d\sigma_3 - \chi_1\,\chi_3\, d\sigma_4\,,\\
-e^{-\varphi_1+\varphi_2+\varphi_3+\varphi_4} *F_{(2)2} &= d\psi_2 - \chi_1\, d\sigma_4\,,\\
-e^{-\varphi_1-\varphi_2+\varphi_3+\varphi_4} *\mathcal{F}^1_{(2)} &= d\psi_3 - \chi_2\, d\psi_2 - \chi_1\, d\sigma_1 + \chi_1\,\chi_2\, d\sigma_4\,,\\
-e^{-\varphi_1-\varphi_2-\varphi_3+\varphi_4} *\mathcal{F}^2_{(2)} &= d\psi_4 + \chi_2\, d\psi_1 - \chi_3\, d\psi_3 - \chi_1\, d\sigma_2 + \chi_2\,\chi_3\, d\psi_2\\
&\quad -\chi_1\,\chi_2\, d\sigma_3 + \chi_1\,\chi_3\, d\sigma_1 - \chi_1\,\chi_2\,\chi_3\, d\sigma_4\,.
\end{aligned} \tag{9}$$

The Lagrangian (7) can be re-expressed as

$$\mathcal{L}_3 = R\,{*\mathbb{1}} - \tfrac{1}{2}{*d\varphi_i} \wedge d\varphi_i - \tfrac{1}{2}\sum_{\alpha=1}^{12} \eta_\alpha\, e^{\vec{a}_\alpha\cdot\vec{\varphi}}\, {*F^\alpha} \wedge F^\alpha\,, \tag{10}$$

where each of the twelve 1-form field strengths F^α can be read off by comparison with the twelve axion kinetic terms in (7). Likewise, the corresponding dilaton vector $\vec{a}_\alpha$ can be read off from the dilatonic prefactor of each axion kinetic term, and the coefficients $\eta_\alpha = \pm 1$ can be read off from the signs of the kinetic terms. One can then easily see that, as in the constructions in [20], we have

$$dF^\alpha = \tfrac{1}{2} f^\alpha{}_{\beta\gamma}\, F^\beta \wedge F^\gamma\,. \tag{11}$$

Introducing generators E_α and defining $\mathcal{F} \equiv F^\alpha\, E_\alpha$, one can express (11) as $d\mathcal{F} = \mathcal{F} \wedge \mathcal{F}$, where

$$[E_\alpha, E_\beta] = f^\gamma{}_{\alpha\beta}\, E_\gamma\,. \tag{12}$$

As in the discussions in [20], the E_α are easily seen to be the positive-root generators of the global symmetry group $O(4,4)$. We also introduce the four Cartan generators $\vec{H}$, which satisfy

$$[\vec{H}, E_\alpha] = \vec{a}_\alpha\, E_\alpha\,. \tag{13}$$

In an obvious notation, we may label the twelve positive-root generators by

$$E_\alpha = (E_{\chi_1}, E_{\chi_2}, \ldots, E_{\sigma_1}, E_{\sigma_2}, \ldots, E_{\psi_1}, E_{\psi_2}, \ldots)\,. \tag{14}$$

The simple root generators are $(E_{\chi_1}, E_{\chi_2}, E_{\chi_3}, E_{\sigma_4})$. The non-vanishing commutators are given by

$$
\begin{aligned}
&[E_{\sigma_1}, E_{\psi_1}] = E_{\chi_4}\,, \quad [E_{\sigma_2}, E_{\psi_2}] = E_{\chi_4}\,, \quad [E_{\sigma_3}, E_{\psi_3}] = E_{\chi_4}\,, \quad [E_{\sigma_4}, E_{\psi_4}] = E_{\chi_4}\,, \\
&[E_{\chi_2}, E_{\sigma_4}] = -E_{\sigma_1}\,, \quad [E_{\chi_2}, E_{\sigma_3}] = E_{\sigma_2}\,, \quad [E_{\chi_3}, E_{\sigma_1}] = -E_{\sigma_2}\,, \quad [E_{\chi_3}, E_{\sigma_4}] = E_{\sigma_3}\,, \\
&[E_{\chi_3}, E_{\psi_2}] = E_{\psi_1}\,, \quad [E_{\chi_1}, E_{\sigma_3}] = -E_{\psi_1}\,, \quad [E_{\chi_1}, E_{\sigma_4}] = -E_{\psi_2}\,, \quad [E_{\chi_2}, E_{\psi_2}] = -E_{\psi_3}\,, \\
&[E_{\chi_1}, E_{\sigma_1}] = -E_{\psi_3}\,, \quad [E_{\chi_2}, E_{\psi_1}] = E_{\psi_4}\,, \quad [E_{\chi_3}, E_{\psi_3}] = -E_{\psi_4}\,, \quad [E_{\chi_1}, E_{\sigma_2}] = -E_{\psi_4}\,.
\end{aligned}
\tag{15}
$$

Following [20] we now define a Borel-gauge coset representative $\mathcal{V}$ as follows:

$$
\mathcal{V} = e^{\frac{1}{2}\vec{\varphi}\cdot\vec{H}}\, \mathcal{U}_\chi\, \mathcal{U}_\sigma\, \mathcal{U}_\psi\,, \tag{16}
$$

where

$$
\begin{aligned}
\mathcal{U}_\chi &= e^{\chi_1\, E_{\chi_1}}\, e^{\chi_2\, E_{\chi_2}}\, e^{\chi_3\, E_{\chi_3}}\, e^{\chi_4\, E_{\chi_4}}\,, \\
\mathcal{U}_\sigma &= e^{\sigma_1\, E_{\sigma_1}}\, e^{\sigma_2\, E_{\sigma_2}}\, e^{\sigma_3\, E_{\sigma_3}}\, e^{\sigma_4\, E_{\sigma_4}}\,, \\
\mathcal{U}_\psi &= e^{\psi_1\, E_{\psi_1}}\, e^{\psi_2\, E_{\psi_2}}\, e^{\psi_3\, E_{\psi_3}}\, e^{\psi_4\, E_{\psi_4}}\,.
\end{aligned}
\tag{17}
$$

A straightforward calculation shows that if we define $\mathcal{U} \equiv \mathcal{U}_\chi\, \mathcal{U}_\sigma\, \mathcal{U}_\psi$ then

$$
d\mathcal{U}\, \mathcal{U}^{-1} = \mathcal{F} = \sum_\alpha F^\alpha\, E_\alpha\,, \tag{18}
$$

and we also have

$$
d\mathcal{V}\, \mathcal{V}^{-1} = \tfrac{1}{2} d\vec{\varphi}\cdot\vec{H} + \sum_\alpha e^{\frac{1}{2}\vec{a}_\alpha\cdot\vec{\varphi}}\, F^\alpha\, E_\alpha\,. \tag{19}
$$

Defining

$$
\mathcal{M} \equiv \mathcal{V}^T\, \eta\, \mathcal{V}\,, \tag{20}
$$

it can be seen that the fully-dualised three-dimensional Lagrangian (7) can be written as

$$
e^{-1}\, \mathcal{L}_3 = R - \tfrac{1}{8}\mathrm{tr}(\partial\mathcal{M}^{-1}\, \partial\mathcal{M})\,. \tag{21}
$$

This makes the $O(4,4)$ global symmetry manifest. Note that the constant matrix η in (20) is chosen so that the required distribution of positive and negative signs in the kinetic terms in (7) is obtained. Specifically, η is preserved under an $O(4,\mathbb{C})$ subgroup of $O(4,4)$ matrices K:

$$
K^T\, \eta\, K = \eta\,. \tag{22}
$$

(If we had instead performed a spacelike reduction to $D = 3$, so that all the kinetic terms were of the standard sign, we would take $\eta = \mathbb{1}$, and the subgroup of $O(4,4)$ matrices satisfying $K^T\, K = \mathbb{1}$ would be $O(4)\times O(4)$.)

In order to generate the 4-charge solution, we shall act on the dimensional reduction of the uncharged Kerr black hole with an $O(1,1)^4$ subgroup of $O(4,4)$. Specifically, we shall take the $O(1,1)^4$ generators to be

$$\lambda_1 = E_{\psi_1} + E_{\psi_1}^T\,, \qquad \lambda_2 = E_{\sigma_2} + E_{\sigma_2}^T\,,$$
$$\lambda_3 = E_{\psi_3} + E_{\psi_3}^T\,, \qquad \lambda_4 = E_{\sigma_4} + E_{\sigma_4}^T\,, \tag{23}$$

and the $O(1,1)^4$ matrix

$$\Lambda \equiv e^{\delta_i\,\lambda_i} \tag{24}$$

will be used to act on $\mathcal{V}$ by right multiplication. In principle, we can calculate the resulting transformations of the fields from

$$\mathcal{V}' = \mathcal{O}\,\mathcal{V}\,\Lambda\,, \tag{25}$$

where $\mathcal{O}$ is an $O(4,\mathbb{C})$ compensating transformation that restores the coset representative to the Borel gauge as in (16). In practice, the drawback to this approach is that finding the required compensating transformation can be rather tricky. Instead, we can calculate the field transformations using

$$\mathcal{M}' = \Lambda^T\,\mathcal{M}\,\Lambda\,, \tag{26}$$

which avoids the need to find the compensating transformation. The price to be paid for this is that $\mathcal{M}$ is a much more complicated matrix than $\mathcal{V}$. However, by using an explicit realisation for the $O(4,4)$ matrices (see appendix A), the problem is easily tractable by computer.

2.2 $O(1,1)^4$ transformation of a reduced uncharged solution

When we implement the solution-generating procedure, our starting point in all cases will be an uncharged four-dimensional solution of the ungauged $\mathcal{N}=2$ supergravity coupled to three vector multiplets, whose bosonic equations of motion are described by the Lagrangian (1). More specifically, in all our examples the starting point will be a solution of pure four-dimensional gravity, i.e. a Ricci-flat metric, admitting a timelike Killing vector $\partial/\partial t$. After reduction on the t direction, it follows that the only non-trivial three-dimensional fields will be the 3-metric ds_3^2, the Kaluza-Klein vector $\mathcal{B}_{(1)}$ and the Kaluza-Klein scalar φ_4. The Kaluza-Klein vector is then dualised to an axion, χ_4, using (8). In view of the fact that all the scalars σ_i and ψ_i (associated with the reduction of the 4 four-dimensional vector fields) are zero in the starting configuration, the dualisation of $\mathcal{B}_{(1)}$ at this stage is therefore simply given by

$$e^{2\varphi_4}\,{*d\mathcal{B}_{(1)}} = d\chi_4\,. \tag{27}$$

We now implement the $O(1,1)^4$ transformations, as described in section 2.1, taking as our starting point a three-dimensional configuration where only ds_3^2, φ_4 and χ_4 are non-trivial. For convenience, we shall denote these starting expressions for φ_4 and χ_4 by $\tilde\varphi_4$ and $\tilde\chi_4$, and then we denote the final expressions for all the $O(1,1)^4$-transformed fields by their symbols without tildes. (Since the three-dimensional metric is inert under $O(4,4)$ transformations, we don't need to introduce a tilde on the starting ds_3^2.)

The starting coset representative $\mathcal{V}$ is therefore given by

$$\mathcal{V} = e^{\frac{1}{2}\tilde\varphi_4\, H_4}\, e^{\tilde\chi_4\, E_{\chi_4}}\,. \tag{28}$$

Constructing $\mathcal{M} = \mathcal{V}^T\,\eta\,\mathcal{V}$, and acting with the $O(1,1)^4$ matrix Λ as in (26), we can obtain the transformed three-dimensional solution. Our results for the three-dimensional fields after $O(1,1)^4$ transformation are as follows:

$$\begin{aligned}
\sigma_1 &= \frac{\tilde\chi_4\Big(h_1(c_{234}s_1 - c_1 s_{234}\, e^{\tilde\varphi_4}) + c_1 s_1 s_{1234}\,\tilde\chi_4^2\Big)}{W^2}\,,\\
\sigma_2 &= \frac{c_2}{s_2} - \frac{c_2 h_1 h_3 h_4 + (c_{134}s_2 - c_2 s_{134}\, e^{\tilde\varphi_4})\, s_{134}\,\tilde\chi_4^2}{s_2 W^2}\,,\\
\sigma_3 &= \frac{\tilde\chi_4\Big(h_3(c_{124}s_3 - c_3 S_{124}\, e^{\tilde\varphi_4}) + c_3 s_3 s_{1234}\,\tilde\chi_4^2\Big)}{W^2}\,,\\
\sigma_4 &= \frac{c_4}{s_3} - \frac{c_4 h_1 h_2 h_3 + (c_{123}s_4 - c_4 s_{123}\, e^{\tilde\varphi_4})\, s_{123}\,\tilde\chi_4^2}{s_4 W^2}\,,\\
\psi_1 &= \frac{c_1}{s_1} - \frac{c_1 h_2 h_3 h_4 + (c_{234}s_1 - c_1 s_{234}\, e^{\tilde\varphi_4})\, s_{234}\,\tilde\chi_4^2}{s_1 W^2}\,,\\
\psi_2 &= -\frac{\tilde\chi_4\Big(h_2(c_{134}s_2 - c_2 s_{134}\, e^{\tilde\varphi_4}) + c_2 s_2 s_{1234}\,\tilde\chi_4^2\Big)}{W^2}\,,\\
\psi_3 &= \frac{c_3}{s_3} - \frac{c_3 h_1 h_2 h_4 + (c_{124}s_3 - c_3 s_{124}\, e^{\tilde\varphi_4})\, s_{124}\,\tilde\chi_4^2}{s_3 W^2}\,,\\
\psi_4 &= -\frac{\tilde\chi_4\Big(h_4(c_{123}s_4 - c_4 s_{123}\, e^{\tilde\varphi_4}) + c_4 s_4 s_{1234}\,\tilde\chi_4^2\Big)}{W^2}\,,\\
e^{\varphi_1} &= \frac{h_1 h_3 + s_{13}^2\tilde\chi_4^2}{W}\,,\qquad e^{\varphi_2} = \frac{h_2 h_3 + s_{23}^2\tilde\chi_4^2}{W}\,,\\
e^{\varphi_3} &= \frac{h_1 h_2 + s_{12}^2\tilde\chi_4^2}{W}\,,\qquad e^{\varphi_4} = \frac{e^{\tilde\varphi_4}}{W}\,,\qquad \chi_1 = \frac{(c_{13}s_{24} - c_{24}s_{13})\tilde\chi_4}{h_1 h_3 + s_{13}^2\tilde\chi_4^2}\,,\\
\chi_2 &= \frac{(c_{14}s_{23} - c_{23}s_{14})\tilde\chi_4}{h_2 h_3 + s_{23}^2\tilde\chi_4^2}\,,\qquad \chi_3 = \frac{(c_{12}s_{34} - c_{34}s_{12})\tilde\chi_4}{h_1 h_2 + s_{12}^2\tilde\chi_4^2}\,,\\
\chi_4 &= \frac{\tilde\chi_4}{W^2(h_1 h_2 + s_{12}^2\tilde\chi_4^2)}\Big\{h_1 h_2\Big[c_{1234}(1+s_2^2+s_4^2) + s_{1234}(1+s_2^2+s_4^2)e^{2\tilde\varphi_4}\\
&\quad -\Big(c_{1234}(s_2^2+s_4^2) + s_{1234}(2+s_2^2+s_4^2)\Big)e^{\tilde\varphi_4}\Big] + s_{1234}s_{12}^2(1+s_2^2+s_4^2)\tilde\chi_4^4\\
&\quad + s_{12}\chi_4^2\Big[c_{12}(c_{12}s_{34} + c_{34}s_{12})(1+s_2^2+s_4^2) - \Big((c_{1234}s_{12}(s_2^2+s_4^2)
\end{aligned}$$

$$
\begin{aligned}
&+s_{34}(s_1^2+s_2^2+5s_{12}^2+s_2^4+3s_1^2s_2^4+s_1^2s_4^2+s_{24}^2+3s_{124}^2)\Big)e^{\tilde\varphi_4}\\
&+2s_{1234}s_{12}(1+s_2^2+s_4^2)e^{2\tilde\varphi_4}\Big]\Big\}\,, \qquad (29)
\end{aligned}
$$

where

$$
\begin{aligned}
h_i &= c_i^2-s_i^2\,e^{\tilde\varphi_4}\,,\quad c_{i_1\cdots i_n}=\cosh\delta_i\,\cdots\,\cosh\delta_{i_n}\,,\quad s_{i_1\cdots i_n}=\sinh\delta_i\,\cdots\,\sinh\delta_{i_n}\,,\\
W^2 &= h_1h_2h_3h_4+\tilde\chi_4^2\Big(2c_{1234}s_{1234}-(s_{123}^2+s_{124}^2+s_{134}^2+s_{234}^2+4s_{1234}^2)\,e^{\tilde\varphi_4}\\
&\qquad +2s_{1234}^2e^{2\tilde\varphi_4}\Big)+s_{1234}\tilde\chi_4^4\,. \qquad (30)
\end{aligned}
$$

3 4-Charge Rotating Black Holes in Ungauged Supergravity

In this section, we implement the procedure described in section 2 to generate the solution for a 4-charge rotating black hole in four dimensions. Our starting point, therefore, is simply the four-dimensional Kerr solution, namely

$$
\begin{aligned}
ds_4^2 &= -dt^2+\frac{2m\,r}{\rho^2}\,(dt-a\sin^2\theta d\varphi)^2+(r^2+a^2)\sin^2\theta d\varphi^2+\rho^2(\frac{dr^2}{\Delta}+d\theta^2)\,,\\
\Delta &= r^2-2mr+a^2\,,\qquad \rho^2=r^2+a^2\cos^2\theta\,. \qquad (31)
\end{aligned}
$$

Recasting it in the form (4), we can read off the reduced three-dimensional metric, Kaluza-Klein vector and Kaluza-Klein scalar:

$$
\begin{aligned}
ds_3^2 &= (\rho^2-2mr)\Big(\frac{dr^2}{\Delta}+d\theta^2\Big)+\Delta\,\sin^2\theta\,d\phi^2\,,\\
\mathcal{B}_{(1)} &= \frac{2m\,a\,r\,\sin^2\theta\,d\phi}{\rho^2-2mr}\,,\qquad e^{\tilde\varphi_4}=1-\frac{2mr}{\rho^2}\,. \qquad (32)
\end{aligned}
$$

After dualisation, using (27), the Kaluza-Klein 1-form $\mathcal{B}_{(1)}$ becomes the axion $\tilde\chi_4$, which is given by

$$
\tilde\chi_4=\frac{2m\,a\,\cos\theta}{\rho^2}\,. \qquad (33)
$$

All the other axions and dilatons in the three-dimensional theory described by (7) are zero. Note that, in line with the notation of section 2.1, we have placed tildes on the starting expressions for the fields $\tilde\varphi_4$ and $\tilde\chi_4$. The post-transformation fields are then written without tildes.

After the $O(1,1)^4$ transformation, the fields are given by (29). Before lifting the solution back to four dimensions, we must dualise the transformed axions ψ_i and χ_4 back to 1-form potentials, so that we can retrace the reduction steps. After performing the dualisations in three dimensions, we find that the dilatons, axions and 1-form potentials are given by

$$
\chi_1 = \frac{2m\,u\,(c_{13}s_{24}-c_{24}s_{13})}{r_1\,r_3+u^2}\,,\qquad \chi_2=\frac{2m\,u\,(c_{14}s_{23}-c_{23}s_{14})}{r_2\,r_3+u^2}\,,
$$

$$
\begin{aligned}
\chi_3 &= \frac{2m\,u\,(c_{12}s_{34}-c_{34}s_{12})}{r_1\,r_2+u^2}\,, \qquad e^{\varphi_1}=\frac{r_1\,r_3+u^2}{W}\,,\\
e^{\varphi_2} &= \frac{r_2\,r_3+u^2}{W}\,, \quad e^{\varphi_3}=\frac{r_1\,r_2+u^2}{W}\,, \quad e^{\varphi_4}=\frac{\rho^2-2mr}{W}\,,\\
\sigma_1 &= \frac{2m\,u}{W^2}\Big[(r\,r_1+u^2)(c_{234}s_1-s_{234}c_1)+2mr_1s_{234}c_1\Big]\,,\\
\sigma_2 &= \frac{1}{W^2}\Big[2m\,c_2s_2\,(r_1\,r_3\,r_4+r\,u^2)+4m^2\,u^2\,e_2\Big]\\
\sigma_3 &= \frac{2m\,u}{W^2}\Big[(r\,r_3+u^2)(c_{124}s_3-s_{124}c_3)+2mr_3s_{124}c_3\Big]\,,\\
\sigma_4 &= \frac{1}{W^2}\Big[2m\,c_4s_4\,(r_1\,r_2\,r_3+r\,u^2)+4m^2\,u^2\,e_4\Big]\,,\\
\mathcal{B}_{(1)} &= \frac{2m(a^2-u^2)(rc_{1234}-(r-2m)s_{1234})}{a(\rho^2-2mr)}\,d\phi\,,\\
A_{(1)1} &= -\frac{2m\,u\,c_1s_1\,(r^2+a^2-2mr)}{a(\rho^2-2mr)}\,d\phi\,,\\
A_{(1)2} &= \frac{2m(a^2-u^2)((r-2m)c_2s_{134}-rc_{134}s_2)}{a(\rho^2-2mr)}\,d\phi\,,\\
\mathcal{A}^1_{(1)} &= -\frac{2m\,u\,c_3s_3\,(r^2+a^2-2m\,r)}{a(\rho^2-2mr)}\,d\phi\,,\\
\mathcal{A}^2_{(1)} &= \frac{2m(a^2-u^2)((r-2m)c_4s_{123}-rc_{123}s_4)}{a(\rho^2-2mr)}\,d\phi\,,
\end{aligned}
\tag{34}
$$

where

$$
\begin{aligned}
W^2 &= r_1\,r_2\,r_3\,r_4+u^4+u^2[2r^2+2mr(s_1^2+s_2^2+s_3^2+s_4^2)\\
&\quad +8m^2c_{1234}s_{1234}-4m^2(s_{123}^2+s_{124}^2+s_{134}^2+s_{234}^2+2s_{1234}^2)]\,,\\
r_i &= r+2ms_i^2\,, \qquad u=a\cos\theta\,,\\
c_{i_1\cdots i_n} &= \cosh\delta_{i_1}\cdots\cosh\delta_{i_n}\,, \qquad s_{i_1\cdots i_n}=\sinh\delta_{i_1}\cdots\sinh\delta_{i_n}\,.
\end{aligned}
\tag{35}
$$

We also define

$$
e_1=c_{234}\,s_{234}\,(c_1^2+s_1^2)-c_1\,s_1\,(s_{23}^2+s_{24}^2+s_{34}^2+2s_{234}^2)\,, \tag{36}
$$

together with analogous expressions for e_2, e_3 and e_4 defined in the obvious way, with the label 2, 3 or 4 singled out in place of the label 1.

3.1 Four-dimensional 4-charge solution

The final step is to lift this three-dimensional solution back to $D=4$, using the Kaluza-Klein reduction rules (4) and (5). Thus the four-dimensional metric for the 4-charge rotating black hole solution is given by

$$
ds_4^2=-\frac{\rho^2-2mr}{W}\,(dt+\mathcal{B}_{(1)})^2+W\,\Big(\frac{dr^2}{\Delta}+d\theta^2+\frac{\Delta\,\sin^2\theta\,d\phi^2}{\rho^2-2mr}\Big)\,. \tag{37}
$$

The 4 four-dimensional gauge potentials, with we denote with hats here to distinguish them from the three-dimensional ones, are given by

$$\begin{aligned}
\hat{A}_{(1)1} &= (A_{(1)1} + \sigma_1\, \mathcal{B}_{(1)}) + \sigma_1\, dt\,,\\
\hat{A}_{(1)2} &= (A_{(1)2} + \sigma_2\, \mathcal{B}_{(1)}) + \sigma_2\, dt\,,\\
\hat{\mathcal{A}}^1_{(1)} &= (\mathcal{A}^1_{(1)} + \sigma_3\, \mathcal{B}_{(1)}) + \sigma_3\, dt\,,\\
\hat{\mathcal{A}}^2_{(1)} &= (\mathcal{A}^2_{(1)} + \sigma_4\, \mathcal{B}_{(1)}) + \sigma_4\, dt\,.
\end{aligned} \tag{38}$$

The field strengths $\hat{F}_{(2)1}$ and $\hat{\mathcal{F}}^1_{(2)}$ carry magnetic charges, while the field strengths $\hat{F}_{(2)2}$ and $\hat{\mathcal{F}}^2_{(2)}$ carry electric charges. The explicit expressions for the electric gauge potentials are

$$\begin{aligned}
\hat{A}_{12} &= \frac{2m}{aW^2}\Big\{(r_1\, r_3\, r_4 + r\, u^2)[c_2\, s_2\, adt - (a^2-u^2)(c_{134}\, s_2 - s_{134}\, c_2)\, d\phi]\\
&\qquad + 2m\, u^2\, [e_2\, adt - (a^2-u^2)\, s_{134}\, c_2 d\phi]\Big\}\,,\\
\hat{\mathcal{A}}^2_1 &= \frac{2m}{aW^2}\Big\{(r_1\, r_2\, r_3 + r\, u^2)[a\, c_4\, s_4\, dt - (a^2-u^2)(c_{123}\, s_4 - s_{123}\, c_4)\, d\phi]\\
&\qquad + 2m\, u^2\, [a\, e_4\, dt - (a^2-u^2) s_{123}\, c_4\, d\phi]\Big\}\,.
\end{aligned} \tag{39}$$

The magnetic gauge potentials are more complicated; they take the form

$$\begin{aligned}
\hat{A}_{(1)1} &= \frac{2m\, u}{aW^2}\Big\{(r\, r_1 + u^2)[(c_{234}\, s_1 - s_{234}\, c_1)\, a\, dt - c_1\, s_1\, a^2\, d\phi] + 2mr_1\, s_{234}\, c_1\, adt\\
&\qquad - \Big(c_1 s_1 (r_1 r_2 r_3 r_4 + u^2[r^2 + 2m\, r(s_2^2 + s_3^2 + s_4^2) - 4m^2 s_{234}^2])\\
&\qquad\quad + 4m^2 u^2 c_{234} s_{234} s_1^2 + 2m e_1 (a^2 r_1 - r\, u^2)\Big) d\phi\Big\}\,,\\
\hat{\mathcal{A}}_{(1)1} &= \frac{2m\, u}{aW^2}\Big\{(r\, r_3 + u^2)[(c_{124}\, s_3 - s_{124}\, c_3)\, a\, dt - c_3\, s_3\, a^2\, d\phi] + 2mr_3\, s_{124}\, c_3\, adt\\
&\qquad - \Big(c_3 s_3 (r_1 r_2 r_3 r_4 + u^2[r^2 + 2m\, r(s_1^2 + s_2^2 + s_4^2) - 4m^2 s_{124}^2])\\
&\qquad\quad + 4m^2 u^2 c_{124} s_{124} s_3^2 + 2m e_3 (a^2 r_3 - r\, u^2)\Big) d\phi\Big\}\,.
\end{aligned} \tag{40}$$

The remaining four-dimensional fields, namely the three dilatons $(\varphi_1, \varphi_2, \varphi_3)$ and the three axions (χ_1, χ_2, χ_3), are simply given by the three-dimensional expressions in (34). The results we have obtained here complete those that were presented in [4], where the explicit form of the four gauge potentials (39) and (40) was not given.

3.2 Four-dimensional solution with pairwise-equal charges

A simple special case arises if we set the two electric charges equal, by taking $\delta_4 = \delta_2$, and also set the two magnetic charges equal, by taking $\delta_3 = \delta_1$. We then find that the previous expressions reduce as follows. The function W becomes $W = r_1\, r_2 + a^2\, \cos^2\theta$, and the

four-dimensional metric is given by

$$\begin{aligned} ds_4^2 &= -\frac{\rho^2-2mr}{r_1\,r_2+a^2\,\cos^2\theta}\Big(dt-\frac{a\,\sin^2\theta\,(r^2-2mr-r_1\,r_2)\,d\phi}{\rho^2-2mr}\Big)^2 \\ &\quad +(r_1\,r_2+a^2\,\cos^2\theta)\Big(\frac{dr^2}{\Delta}+d\theta^2+\frac{\Delta\,\sin^2\theta\,d\phi^2}{\rho^2-2mr}\Big)\,. \end{aligned} \tag{41}$$

The remaining four-dimensional fields are given by

$$\begin{aligned} e^{\varphi_1} &= \frac{r_1^2+a^2\,\cos^2\theta}{r_1\,r_2+a^2\,\cos^2\theta}\,, \qquad \chi_1 = \frac{a\,(r_2-r_1)\,\cos\theta}{r_1^2+a^2\,\cos^2\theta}=\frac{2m\,a\,(s_2^2-s_1^2)\,\cos\theta}{r_1^2+a^2\,\cos^2\theta}\,, \\ \hat{A}_{(1)1} &= \hat{\mathcal{A}}^1_{(1)}=\frac{2m\,s_1\,c_1\,[a\,dt-(r_1\,r_2+a^2)\,d\phi]\,\cos\theta}{r_1\,r_2+a^2\,\cos^2\theta}\,, \\ \hat{A}_{(1)2} &= \hat{\mathcal{A}}^2_{(1)}=\frac{2m\,s_2\,c_2\,r_1\,[dt-a\,\sin^2\theta\,d\phi]}{r_1\,r_2+a^2\,\cos^2\theta}\,, \\ \varphi_2 &= \varphi_3=\chi_2=\chi_3=0\,. \end{aligned} \tag{42}$$

It can easily be verified that if one additionally sets $\delta_1=\delta_2$, the solution reduces, as expected, to a dyonic Kerr-Newman solution of the pure Einstein-Maxwell system, with equal electric and magnetic charges.

It is useful to write down the truncated four-dimensional theory for which the 2-charge configuration presented above is a solution. The truncation is performed by setting the field strengths in (1) pairwise equal (according to the scheme in (42), and at the same time setting $\varphi_2=\varphi_3=\chi_2=\chi_3=0$. It is easily verified that this is a consistent truncation of the full equations of motion. The Lagrangian for the truncated system is

$$\begin{aligned} \mathcal{L}_4 &= R\,{*\mathbb{1}}-\tfrac12{*d\varphi_1}\wedge d\varphi_1-\tfrac12 e^{2\varphi_1}\,{*d\chi_1}\wedge d\chi_1-\tfrac12 e^{-\varphi_1}\,({*F_{(2)1}}\wedge F_{(2)1}+{*F_{(2)2}}\wedge F_{(2)2}) \\ &\quad -\tfrac12\chi_1\,(F_{(2)1}\wedge F_{(2)1}+F_{(2)2}\wedge F_{(2)2})\,, \end{aligned} \tag{43}$$

where, having equated the pairs of field strengths, we have then rescaled them by factors of $1/\sqrt{2}$ in order to restore the canonical normalisation in (43).

We can present the 2-charge solution in (42) in a more elegant form. Recalling that we now have

$$W=r_1\,r_2+a^2\,\cos^2\theta\,, \tag{44}$$

the terms in the metric can be reorganised so that it becomes

$$ds_4^2=-\frac{\Delta}{W}\,(dt-a\,\sin^2\theta\,d\phi)^2+W\Big(\frac{dr^2}{\Delta}+d\theta^2\Big)+\frac{\sin^2\theta}{W}\,[a\,dt-(r_1r_2+a^2)d\phi]^2\,. \tag{45}$$

The remaining fields in the 2-charge solution may be written as

$$e^{\varphi_1} = 1+\frac{r_1\,(r_1-r_2)}{W}=\frac{r_1^2+a^2\,\cos^2\theta}{r_1\,r_2+a^2\,\cos^2\theta}\,, \qquad \chi_1=\frac{a\,(r_2-r_1)\,\cos\theta}{r_1^2+a^2\,\cos^2\theta}\,,$$

$$
\begin{aligned}
A_{(1)1} &= \frac{2\sqrt{2}\, m\, s_1\, c_1\, [a\, dt - (r_1 r_2 + a^2) d\phi]\, \cos\theta}{W}\,, \\
A_{(1)2} &= \frac{2\sqrt{2} m\, s_2\, c_2\, r_1\, (dt - a\, \sin^2\theta\, d\phi)}{W}\,.
\end{aligned} \tag{46}
$$

4 Charged Rotating Black Holes in Gauged Supergravity

In this section, we look for generalisations of the charged rotating black holes to the case of *gauged* four-dimensional supergravity. Since there is no longer a solution-generating technique for deriving the solutions in the gauged theories, we instead resort to a technique of "inspired guesswork," followed by brute-force verification that the equations of motion are satisfied. The verification is purely mechanical, but the process of guessing, or making an ansatz, for the form of the gauged solution is not so straightforward. In fact, so far we have succeeded in guessing the form of the gauged solution only in the case that the four charges are set pairwise equal. Thus in this section, we shall present our results for the gauged generalisation of the pairwise-equal ungauged solutions obtained in section (3.2).

The easiest way to discuss the solution is by simply augmenting the bosonic Lagrangian (1) by the subtraction of the scalar potential that arises in the gauged supergravity. For the discussion of the solutions with pairwise-equal charges, which we are considering here, we can take the scalar potential to be that of the $\mathcal{N}=4$ gauged $SO(4)$ theory, namely

$$
V = -g^2 \sum_{i=1}^{3} (2\cosh\varphi_i + \chi_i^2\, e^{\varphi_i})\,. \tag{47}
$$

with $\varphi_2 = \varphi_3 = \chi_2 = \chi_3 = 0$. The two electromagnetic charges in our pairwise-equal solution will then be carried by fields in $U(1)$ subgroups of the two $SU(2)$ factors in $SO(4) \sim SU(2)\times SU(2)$. Thus we may consider the bosonic Lagrangian

$$
\begin{aligned}
\mathcal{L}_4 &= R\,{*\mathbb{1}} - \tfrac12 {*d\varphi_1}\wedge d\varphi_1 - \tfrac12 e^{2\varphi_1}\, {*d\chi_1}\wedge d\chi_1 - \tfrac12 e^{-\varphi_1}\, ({*F_{(2)1}}\wedge F_{(2)1} + {*F_{(2)2}}\wedge F_{(2)2}) \\
&\quad -\tfrac12 \chi_1\, (F_{(2)1}\wedge F_{(2)1} + F_{(2)2}\wedge F_{(2)2}) - g^2\, (4 + 2\cosh\varphi_1 + e^{\varphi_1}\, \chi_1^2)\, {*\mathbb{1}}\,.
\end{aligned} \tag{48}
$$

It should be emphasised that the Lagrangian (48) is *not*, as it stands, the bosonic sector of any supergravity theory. First of all, we have included only the $U(1)\times U(1)$ subset of the $SU(2)\times SU(2)$ gauge fields of $SO(4)$-gauged $\mathcal{N}=4$ supergravity. This abelian truncation is consistent as a bosonic trunction, but not as a supersymmetric truncation. Secondly, even if we included all the $SU(2)\times SU(2)$ gauge fields in (48), the Lagrangian would still not be the bosonic sector of the $SO(4)$-gauged $\mathcal{N}=4$ supergravity. The gauge fields of one of the $SU(2)$ factors would have had to have been dualised prior to turning on

the gauging, in order to get a supersymmetrisable theory. Since bare potentials appear in the expressions for the field strengths in the non-abelian gauged theory, dualisation can no longer be performed.

The upshot of the above discussion is that for the purposes of conjecturing, and then verifying, a solution of the gauged theory, it suffices to work with the generalisations of the ungauged solutions (42), and look at the equations of motion following from Lagrangian (48). Having successfully obtained charged rotating black-hole solutions, we can, if we wish, dualise one of the two field strengths. In this dualised form, the black hole can be directly viewed as a solution within $SO(4)$-gauged $\mathcal{N}=4$ supergravity, with the non-zero gauge fields of the solution residing within a $U(1)\times U(1)$ subgroup of $SU(2)\times SU(2)$. It is this dualised formulation that one would need to use if one wanted to test the supersymmetry of the solution.

By studying the form of the known Kerr-Newman-AdS black hole, as well that of the pairwise-equal charge solution (42), we have been able to conjecture the form of the rotating black hole solution of gauged supergravity with two pair-wise equal charges. Verifying the correctness of the conjecture is then a mechanical procedure, which we have performed using Mathematica. Our solution takes the form

$$
\begin{aligned}
ds_4^2 &= -\frac{\Delta_r}{W}\,(dt - a\,\sin^2\theta\,d\phi)^2 + W\left(\frac{dr^2}{\Delta_r}+\frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta\,\sin^2\theta}{W}\,[a\,dt-(r_1r_2+a^2)d\phi]^2\,,\\
e^{\varphi_1} &= = \frac{r_1^2+a^2\cos^2\theta}{W} = 1+\frac{r_1\,(r_1-r_2)}{W}\,,\qquad \chi_1 = \frac{a\,(r_2-r_1)\,\cos\theta}{r_1^2+a^2\,\cos^2\theta}\,,\\
A_{(1)1} &= \frac{2\sqrt{2}\,m\,s_1\,c_1\,[a\,dt-(r_1r_2+a^2)d\phi]\,\cos\theta}{W}\,,\\
A_{(1)2} &= \frac{2\sqrt{2}m\,s_2\,c_2\,r_1\,(dt-a\,\sin^2\theta\,d\phi)}{W}\,,
\end{aligned}
\tag{49}
$$

where r_1 and r_2 are defined in (35), and

$$
\begin{aligned}
\Delta_r &\equiv \Delta + g^2\,r_1\,r_2\,(r_1\,r_2+a^2) = r^2+a^2-2m\,r+g^2\,r_1\,r_2\,(r_1\,r_2+a^2)\,,\\
\Delta_\theta &\equiv 1-g^2\,a^2\,\cos^2\theta\,,\qquad W = r_1\,r_2+a^2\cos^2\theta\,.
\end{aligned}
\tag{50}
$$

Note that the dilaton, axion and gauge potentials are identical to those of the ungauged theory, appearing in (46).

As we mentioned above, it is useful for some purposes to "undualise" the bosonic theory described by (48), so that it is expressed in terms of the fields that arise in the complete $SO(4)$-gauged $\mathcal{N}=4$ supergravity including the fermions. We can do this by introducing a Lagrange multiplier $B_{(1)}$ to enforce the Bianchi identity $dF_{(2)1}=0$, adding the term

$dB_{(1)} \wedge F_{(2)1}$ to the Lagrangian (48), and using the algebraic equation of motion for $F_{(2)1}$ to eliminate $F_{(2)1}$ in favour of $G_{(2)} = dB_{(1)}$. Thus we find

$$e^{-\varphi_1} {*F_{(2)1}} + \chi_1\, F_{(2)1} = G_{(2)}\,, \tag{51}$$

and hence the Lagrangian becomes

$$\begin{aligned}
\mathcal{L}_4 \;=\;& R{*1\!\!1} - \tfrac{1}{2}{*d\varphi_1} \wedge d\varphi_1 - \tfrac{1}{2}e^{2\varphi_1}\, {*d\chi_1} \wedge d\chi_1 - \tfrac{1}{2}e^{-\varphi_1}\, {*F_{(2)2}} \wedge F_{(2)2} - \tfrac{1}{2}\chi_1\, F_{(2)2} \wedge F_{(2)2} \\
& - \frac{1}{2(1+\chi_1^2\, e^{2\varphi_1})}\, (e^{\varphi_1}\, {*G_{(2)}} \wedge G_{(2)} - e^{2\varphi_1}\, \chi_1\, G_{(2)} \wedge G_{(2)}) \\
& - g^2\, (4 + 2\cosh\varphi_1 + e^{\varphi_1}\, \chi_1^2)\, {*1\!\!1}\,.
\end{aligned} \tag{52}$$

In terms of this "undualised" formulation, the 2-charge solution (49) is identical except that instead of having $A_{(1)1}$ which carries magnetic charge, we have

$$B_{(1)} = \frac{2\sqrt{2}m\, s_1\, c_1\, r_2\, (dt - a\, \sin^2\theta\, d\phi)}{W}\,, \tag{53}$$

which carries electric charge.

It is easy to see that when $\delta_1 = \delta_2$, the dilaton ϕ_1 and the axion χ_1 in (49) vanish, and the solution reduces to the AdS-Kerr-Newman black hole with purely electric charge [18, 19]. If instead we set the rotation parameter a to zero, the solution becomes the static AdS black hole constructed in [21], with the four charges set pairwise equal.

It was shown in [22] that the $\mathcal{N} = 4$ gauged supergravity (48) can be obtained from S^7 reduction of eleven-dimensional supergravity, where explicit reduction ansatz were given. We can use the reduction ansatz to lift the charged rotating solution back to $D = 11$. The metric becomes

$$\begin{aligned}
ds_{11}^2 \;=\;& (X^2\cos^2\xi + \sin^2\xi)^{\frac{1}{3}}\, (\widetilde{X}^2\sin^2\xi + \cos^2\xi)^{\frac{1}{3}} \Big\{ -\frac{\Delta_r}{W}\, (dt - a\, \sin^2\theta\, d\phi)^2 \\
& + W\Big(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\Big) + \frac{\Delta_\theta\, \sin^2\theta}{W}\, [a\, dt - (r_1 r_2 + a^2) d\phi]^2 + 4g^{-2}\, d\xi^2 \\
& + \frac{\cos^2\xi [d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2 + (d\psi_1 + \cos\theta_1\, d\phi_1 - 2m\, g\, s_2\, c_2\, r_1 W^{-1} (dt - a\sin^2\theta\, d\phi)^2]}{g^2 (X^2\cos^2\xi + \sin^2\xi)} \\
& + \frac{\sin^2\xi [d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2 + (d\psi_2 + \cos\theta_2\, d\phi_2 - 2m\, g\, s_1\, c_1\, r_2 W^{-1} (dt - a\sin^2\theta\, d\phi)^2]}{g^2 (\widetilde{X}^2\sin^2\xi + \cos^2\xi)} \Big\}\,,
\end{aligned} \tag{54}$$

where

$$X^2 = \frac{r_1^2 + a^2\cos^2\theta}{W}\,, \qquad \widetilde{X}^2 = \frac{r_2^2 + a^2\cos^2\theta}{W}\,, \tag{55}$$

and $(\theta_1, \phi_1, \psi_1)$ and $(\theta_2, \phi_2, \psi_2)$ are Euler angles on the two S^3 factors of the foliation of S^7. The eleven-dimensional metric describes a rotating M2-brane with rotations in both

the world-volume and the transverse space. The 4-form field strength can be easily read off from the reduction formulae in [22]. If we set the world-volume rotation parameter a to zero, the eleven-dimensional solution reduces to the one obtained in [23], where the four transverse angular momenta are set pairwise equal.

5 Generalisations with a NUT Parameter

The construction of four-dimensional charged solutions that we have been discussing so far were obtained by starting from the uncharged Kerr solution, and performing an $O(1,1)^4 \subset O(4,4)$ transformation in a dimensional reduction to three dimensions. Lifting back to four dimensions, this yields a solution of ungauged supergravity with four independent charges, supported by four abelian field strengths. In the case where the four charges were set pairwise equal, we were able to conjecture the generalisation of this solution to gauged supergravity, and to confirm that indeed the equations of motion were satisfied.

More generally, we can start from an uncharged solution where not only the rotation parameter, but also the NUT charge parameter, is non-zero. Reducing this Ricci-flat Kerr-Taub-NUT solution to $D=3$, performing the $O(1,1)^4$ transformation, and lifting back to $D=4$ yields a new solution of ungauged supergravity, with four electromagnetic charges, rotation, mass and NUT charge. Again, in the case that the four electromagnetic charges are set pairwise equal, we can successfully conjecture a generalisation of the solution to gauged supergravity, and verify that indeed it solves the equations.

The procedure for first constructing the generalisations of the Kerr-Taub-NUT metrics to include 4 charges is analogous to the one we followed previously for the Kerr metric. A convenient way of writing the Kerr-Taub-NUT metric, which forms the starting point for our procedure, is in the formulation of Plebanski [24]:

$$ds_4^2 = -\frac{\widetilde{\Delta}_r}{a^2(r^2+u^2)}\,[a dt + u^2\,d\phi]^2 + \frac{\widetilde{\Delta}_u}{a^2(r^2+u^2)}\,[a dt - r^2\,d\phi]^2 + (r^2+u^2)\Big(\frac{dr^2}{\widetilde{\Delta}_r} + \frac{du^2}{\widetilde{\Delta}_u}\Big)\,, \qquad (56)$$

where the functions $\widetilde{\Delta}_r$ and $\widetilde{\Delta}_u$ are given by

$$\widetilde{\Delta}_r = a^2 + r^2 - 2mr\,, \qquad \widetilde{\Delta}_u = a^2 - u^2 + 2\ell\, r\,. \qquad (57)$$

Here m is the mass, ℓ is the NUT parameter, and a is the rotation parameter.

We then implement the reduction (on the time coordinate t) to three dimensions, and performing the $O(1,1)^4$ rotation as described in section (2). After "undualising" scalars to vectors, we lift back to four dimensions, thereby obtaining a solution with four electromagnetic charges. As in our earlier discussion for the Kerr case, two charges, associated with

the $O(1,1)^4$ parameters δ_1 and δ_3, are magnetic, while the two associated with δ_2 and δ_4 are electric.

In the present section, we give our results for the case where we set the charges pairwise equal, i.e. $\delta_1 = \delta_3$ and $\delta_2 = \delta_4$. Since the "inspired guesswork" that leads to the generalisation of the solution to the gauged theory is very similar to that which we used in section (4) for the case with no NUT parameter, we shall in fact directly present our results for the gauged case. We find that the metric of the charged four-dimensional solution is given by

$$ds_4^2 = -\frac{\Delta_r}{a^2\,W}\,[adt + u_1\,u_2\,d\phi]^2 + \frac{\Delta_u}{a^2\,W}\,[adt - r_1\,r_2\,d\phi]^2 + W\,\Big(\frac{dr^2}{\Delta_r} + \frac{du^2}{\Delta_u}\Big)\,, \tag{58}$$

where

$$\begin{aligned} \Delta_r &= \widetilde{\Delta}_r + g^2\,r_1\,r_2\,(r_1\,r_2 + a^2)\,, \qquad \Delta_u = \widetilde{\Delta}_u + g^2\,u_1\,u_2\,(u_1\,u_2 - a^2)\,, \\ W &= r_1\,r_2 + u_1\,u_2\,, \qquad r_i = r + 2m\,s_i^2\,, \qquad u_i = u + 2\ell\,s_i^2\,, \end{aligned} \tag{59}$$

and $s_i = \sinh\delta_i$. The remaining fields are given by

$$\begin{aligned} e^{\varphi_1} &= \frac{r_1^2 + u_1^2}{W}\,, \qquad \chi_1 = \frac{2(s_1^2 - s_2^2)\,(\ell\,r - m\,u)}{r_1^2 + u_1^2}\,, \\ A_{(1)1} &= \frac{2\sqrt{2}s_1\,c_1}{a\,W}\,\Big\{m\,u_1\,[adt - r_1\,r_2\,d\phi] - \ell\,r_1\,[adt + u_1\,u_2\,d\phi]\Big\}\,, \\ A_{(1)2} &= \frac{2\sqrt{2}s_2\,c_2}{a\,W}\,\Big\{\ell\,u_1\,[adt - r_1\,r_2\,d\phi] + m\,r_1\,[adt + u_1\,u_2\,d\phi]\Big\}\,. \end{aligned} \tag{60}$$

The ungauged case is, of course, obtained by setting $g = 0$. The verification of our conjectured result for $g \neq 0$ is straightforward; we used Mathematica to check that the equations of motion following from (52) are indeed satisfied. In Appendix B, we give our complete results for the rotating NUT solutions with four independent charges, in the ungauged case.

It can easily be seen that this solution reduces to the previous one given in (49) if one sets the NUT parameter ℓ to zero. If instead the two charge parameters are set equal, $\delta_1 = \delta_2$, then one has $\varphi_1 = \chi_1 = 0$ and the solution reduces to the charged AdS-Kerr-Taub-NUT solution of Einstein-Maxwell theory with a cosmological constant, as given in [24].

As in the case of the rotating black hole in the previous chapter, we can lift the solution back to $D = 11$, following the reduction ansatz given in [22].

6 Generalisation with Acceleration

In addition to the inclusion of a NUT parameter, there exist more general four-dimensional type-D metrics which include an acceleration parameter as well as mass, NUT charge and

rotation. These metrics can be elegantly described within the framework of Plebanski and Demianski [25], where they are written as

$$ds^2 = \frac{1}{(1-ru)^2}\Big\{\frac{\Delta_u}{r^2+u^2}(dt-r^2d\phi)^2 - \frac{\Delta_r}{r^2+u^2}(dt+u^2d\phi)^2 + (r^2+u^2)(\frac{dr^2}{\Delta_r}+\frac{du^2}{\Delta_u})\Big\}\,. \quad (61)$$

Here

$$\begin{aligned}\Delta_r &= (\gamma-\tfrac{1}{6}\Lambda) - 2m\,r + \epsilon\,r^2 - 2\ell\,r^3 - (\gamma+\tfrac{1}{6}\Lambda)r^4\,,\\ \Delta_u &= (\gamma-\tfrac{1}{6}\Lambda) + 2\ell\,u - \epsilon\,u^2 + 2m\,u^3 - (\gamma+\tfrac{1}{6}\Lambda)u^4\,. \end{aligned} \quad (62)$$

We have included here the cosmological constant Λ; in order to implement the charge-generating procedure of section (2), we shall, of course, need to set $\Lambda = 0$. In fact the general Plebanski-Demianski metrics also include electric and magnetic charges, giving solutions of the Einstein-Maxwell equations. Our interest is in constructing more general solutions in four-dimensional supergravities, along the lines we have considered in previous sections. Thus for our purposes we can begin with the uncharged Plebanski-Demianski metrics. We also turn off the cosmological constant Λ, so that we can apply the solution-generating technique via $O(1,1)^4$ transformations after reduction to three spatial dimensions.

We then perform the $O(1,1)^4$ boosts in three dimensions, undualise the scalars that originally corresponded to vectors, and lift the solution back to $D = 4$, where we can obtain 4-charged rotating and accelerating solutions with mass and NUT charge. We have worked out the solutions with four independent charges, but they are rather involved and so, for simplicity, we shall present here only the solution where charges are set pairwise equal, namely $\delta_3 = \delta_1$ and $\delta_4 = \delta_2$. We find that the solution is given by

$$\begin{aligned}ds^2 &= \frac{1}{(1-ru)^2}\Big\{\frac{\Delta_u}{W}(dt-r_1r_2d\phi)^2 - \frac{\Delta_r}{W}(dt+u_1u_2d\phi)^2 + W(\frac{dr^2}{\Delta_r}+\frac{du^2}{\Delta_u})\Big\}\,,\\ e^{\phi_1} &= \frac{r_1^2+u_1^2}{W}\,, \qquad \chi_1 = \frac{(s_1^2-s_2^2)(r_1u_s-u_1r_s)}{r_1^2+u_1^2}\,,\\ A^1_{(1)} &= \frac{s_1c_1}{(1-ru)^2\,W}\Big\{f_1(dt-r_1r_2d\phi) + f_2(dt+u_1u_2d\phi)\Big\}\,,\\ A^2_{(1)} &= \frac{s_2c_2}{(1-ru)^2W}\Big\{u_1r_s(dt-r_1r_2d\phi) + r_1r_s(dt+u_1u_2d\phi)\Big\}\,, \end{aligned} \quad (63)$$

where

$$\begin{aligned}r_i &= r + r_s\,s_i^2\,, \qquad u_i = u + u_s\,s_i^2\,, \qquad W = r_1r_2 + u_1u_2\,,\\ r_s &= \frac{1}{r}(r^2+\gamma-\frac{\Delta_r}{(1-ru)^2})\,, \qquad u_s = \frac{1}{u}(u^2-\gamma+\frac{\Delta_u}{(1-ru)^2})\,,\\ \Delta_r &= \gamma - 2m\,r + \epsilon\,r^2 - 2\ell\,r^3 - \gamma\,r^4\,,\\ \Delta_u &= \gamma + 2\ell\,u - \epsilon\,u^2 + 2mu^3 - \gamma\,u^4\,, \end{aligned}$$

$$
\begin{aligned}
f_1 &= \gamma\, r^2 - 2(m - \ell\, r^2)u + (\gamma + 4m\, r - \epsilon\, r^2)u^2 - 2\gamma\, r\, u^3\,, \\
f_2 &= \gamma\, u^2 + 2(\ell - m\, u^2)r + (\gamma - 4\ell\, u + \epsilon\, u^2)r^2 - 2\gamma\, u\, r^3\,.
\end{aligned} \tag{64}
$$

In our previous examples, of charged generalisations of the Kerr and Kerr-Taub-NUT metrics, we found that the necessary modifications in order to obtain solutions within *gauged* supergravity required changing only the metric, simply by adding terms to the functions Δ_r and Δ_u. The dilaton, the axion and the two gauge potentials remained unchanged. In the present case, however, these fields would also have to be modified. To see this, we can examine the equations of motion for the vector potentials.

We could only expect that the gauge potentials would require no modification when seeking solutions in the gauged supergravity if the gauge-field equations of motion did not involve the explicit appearance of the functions Δ_r and Δ_u that themselves are modified in the gauged case. In order for the Δ_r and Δ_u functions to be absent in the gauge-field equations, it is necessary that the field strengths take the form

$$
F_{(2)} = \alpha_1 dr \wedge (dt + u_1 u_2 d\phi) + \alpha_2 du \wedge (dt - r_1 r_2 d\phi)\,, \tag{65}
$$

where α_1 and α_2 are functions of r and u. For gauge potential of the form $A_{(1)} = \beta_1 dt + \beta_2 d\phi$, we must then have the conditions

$$
\frac{\partial \beta_2}{\partial r} - u_1 u_2 \frac{\partial \beta_1}{\partial r} = 0\,, \qquad \frac{\partial \beta_2}{\partial u} + r_1 r_2 \frac{\partial \beta_1}{\partial u} = 0\,. \tag{66}
$$

It is straightforward to verify that the above conditions are satisfied by the previous examples (charged generalisations of Kerr or Kerr-Taub-NUT), and consequently, the vector potentials receive no modification when the solutions are generalised to gauged supergravity, which is achieved by adding terms only to Δ_r and Δ_u. In the present example, however, the conditions (66) are not satisfied, and so we expect that the generalisation to gauged supergravity would require modifications to the gauge potentials and, possibly, to the dilaton and axion, as well as to the metric.

7 Conclusions

In this paper we have constructed new charged rotating solutions of four-dimensional ungauged and gauged supergravities. Our new ungauged solutions can be viewed as being embedded within $\mathcal{N} = 2$ supergravity coupled to three vector multiplets. This is itself, of course, embedded within $\mathcal{N} = 8$ supergravity. We first constructed the general ungauged rotating black holes with four independent charges; this completed the presentation

of the solutions first given in [4], which did not give the expressions for the gauge potentials. We did this by employing a solution-generating technique, which involved reducing the four-dimensional theory on the time direction and then acting with global symmetry generators $O(1,1)^4 \subset O(4,4)$ to introduce the charges. We were able also to construct four-dimensional charged generalisations of the Kerr-Taub-NUT solution, and the further generalisation where the acceleration parameter is included too. In all cases we constructed the solutions with four independent charge parameters, although in the case of the rotating metrics with acceleration parameter as well as the NUT parameter, the general 4-charge solutions were rather too complicated to include in the paper. In that case, we therefore only presented the specialisation where the four charges are set pairwise equal.

For the case of the charged rotating black holes, both with and without the inclusion of the NUT parameter, we were able to conjecture the generalisations of the above ungauged solutions to the case of gauged supergravity, after making the specialisation that the four charges are set pairwise equal. We verified the correctness of our conjectured solutions by explicitly confirming that all the equations of motion are satisfied. These solutions are most appropriately viewed as being embedded in $SO(4)$-gauged $\mathcal{N}=4$ supergravity.

The four-dimensional charged rotating black hole solutions that we obtained in this paper provide new gravitational backgrounds for four-dimensional vacua in compactified string theory. In particular, the non-extreme black hole solutions of gauged supergravity provide asymptotically AdS backgrounds that are characterised by their mass, angular momentum and two pair-wise equal charges (implying that they can be viewed as solutions in $\mathcal{N}=4$ gauged supergravity). The more general gauged solutions that we also obtained, where additionally the NUT parameter is non-zero, should provide new information on the dual three-dimensional conformal field theory at non-zero temperature.

Acknowledgments

M.C. and C.N.P. would like to thank CERN for hospitality and support during the initial stages of the work.

A Matrix Realisation of the $O(4,4)$ Generators

The $O(4,4)$ algebra has an $SL(4,\mathbb{R})$ subalgebra, whose positive-root generators are $E_i{}^j$, satisfying

$$[E_i{}^j, E_k{}^\ell] = \delta_k^j\, E_i{}^\ell - \delta_i^\ell\, E_k{}^j\,. \tag{67}$$

To this we append the generators V^{ij}, antisymmetric in i and j, which satisfy the commutation relations

$$[E_i{}^j, V^{k\ell}] = -\delta_i^k\, V^{j\ell} - \delta_i^\ell\, V^{kj}\,, \qquad [V^{ij}, V^{k\ell}] = 0\,. \tag{68}$$

Together, $(E_i{}^j, V^{ij})$ form the positive-root generators of $O(4,4)$. The simple-root generators are

$$V^{12}\,, \qquad E_1{}^2, \qquad E_2{}^3\,, \qquad E_3{}^4\,. \tag{69}$$

In terms of our generators in (14), we have

$$E_{\chi_1} = V^{12}\,, \qquad E_{\chi_2} = E_1{}^2\,, \qquad E_{\chi_3} = E_3{}^4\,, \qquad E_{\sigma_4} = E_2{}^3\,. \tag{70}$$

The remaining positive-root generators in section xxx are given by

$$\begin{aligned} &E_{\sigma_1} = -E_1{}^3\,, \qquad E_{\sigma_2} = -E_1{}^4\,, \qquad E_{\sigma_3} = -E_2{}^4\,, \qquad E_{\chi_4} = V^{34}\,, \\ &E_{\psi_1} = V^{14}\,, \qquad E_{\psi_2} = -V^{13}\,, \qquad E_{\psi_3} = -V^{23}\,, \qquad E_{\psi_4} = -V^{24}\,. \end{aligned} \tag{71}$$

To give an explicit 8×8 matrix realisation, we first introduce the 4×4 matrix e_{ij} which has zeros everywhere except for a 1 at row i, column j. We can then write

$$E_i{}^j = \left(\begin{array}{c|c} -e_{ji} & 0 \\ \hline 0 & e_{ij} \end{array}\right)\,, \qquad V^{ij} = \left(\begin{array}{c|c} 0 & e_{ij} - e_{ji} \\ \hline 0 & 0 \end{array}\right)\,, \tag{72}$$

where each block represents a 4×4 matrix. The four Cartan generators are then represented by

$$\begin{aligned} &H_1 = \mathrm{diag}(1,1,0,0,-1,-1,0,0)\,, \qquad H_2 = \mathrm{diag}(-1,1,0,0,1,-1,0,0)\,, \\ &H_3 = \mathrm{diag}(0,0,-1,1,0,0,1,-1)\,, \qquad H_4 = \mathrm{diag}(0,0,-1,-1,0,0,1,1)\,. \end{aligned} \tag{73}$$

The negative-root generators are given by the matrix transposes of the positive-root generators.

All of the above discussion generalises straightforwardly to $O(n,n)$, with its $SL(n,\mathbb{R})$ subgroup having positive-root generators $E_i{}^j$ $(i<j)$, and the full $O(n,n)$ obtained by augmenting these with V^{ij}. The algebra of the $E_i{}^j$ and V^{ij} is given by (67) and (68), with $1 \le i \le n$. The simple-root generators of $O(n,n)$ are the simple-root generators $E_i{}^{i+1}$ of $SL(n,\mathbb{R})$, augmented by V^{12}.

The matrix η, used in defining $\mathcal{M}$ in (20), as a coset representative for $O(4,4)/O(4,\mathbb{C})$, is given in this basis by

$$\eta = \mathrm{diag}(1,1,-1,-1,1,1,-1,-1)\,. \tag{74}$$

The $O(4,\mathbb{C})$ subgroup comprises $O(4,4)$ matrices K that satisfy $K^T\,\eta\,K=\eta$, and so the $O(4,\mathbb{C})$ generators T_α comprise the subset of $O(4,4)$ generators that satisfy $T_\alpha^T\,\eta+\eta\,T_\alpha=0$. It is straightforward to see that these are

$$X_i=E_{\sigma_i}+E_{\sigma_i}^T\,,\qquad Y_i=E_{\psi_i}+E_{\psi_i}^T\,,\qquad Z_i=E_{\chi_i}-E_{\chi_i}^T\,,\tag{75}$$

where $1\le i\le 4$. The generators X_i and Y_i are non-compact, whilst the generators Z_i are compact.

It is convenient to take the Cartan generators h_i to be combinations of the four generators $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ that were defined in (23), which we exponentiated with parameters δ_i to give the $O(1,1)^4$ charge-generating transformations. In the notation of (75), we have

$$\lambda_1=Y_1\,,\qquad \lambda_2=X_2\,,\qquad \lambda_3=Y_3\,,\qquad \lambda_4=X_4\,.\tag{76}$$

We define

$$\begin{aligned} h_1&=\tfrac12(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\,,\qquad & h_2&=\tfrac12(\lambda_1-\lambda_2-\lambda_3+\lambda_4)\,,\\ h_3&=\tfrac12(\lambda_1-\lambda_2+\lambda_3-\lambda_4)\,,\qquad & h_4&=\tfrac12(\lambda_1+\lambda_2-\lambda_3-\lambda_4)\,. \end{aligned}\tag{77}$$

A straightforward calculation shows that the positive and negative root generators of $O(4,\mathbb{C})$ are then given by

$$E_i^\pm=U_i\pm V_i\,,\tag{78}$$

where

$$\begin{aligned} U_1&=X_1+X_3-Y_2-Y_4\,,\qquad & V_1&=-Z_1+Z_2-Z_3-Z_4\,,\\ U_2&=X_1-X_3+Y_2-Y_4\,,\qquad & V_2&=-Z_1-Z_2-Z_3+Z_4\,,\\ U_3&=X_1-X_3-Y_2+Y_4\,,\qquad & V_3&=Z_1-Z_2-Z_3-Z_4\,,\\ U_4&=X_1+X_3+Y_2+Y_4\,,\qquad & V_4&=Z_1+Z_2-Z_3+Z_4\,. \end{aligned}\tag{79}$$

Under $\vec h\equiv(h_1,h_2,h_3,h_4)$, the roots have weights given by $[\vec h,E_i^\pm]=\pm\vec\alpha_i\,E_i^\pm$, where

$$\vec\alpha_1=(2,0,0,0)\,,\quad \vec\alpha_2=(0,2,0,0)\,,\quad \vec\alpha_3=(0,0,2,0)\,,\quad \vec\alpha_4=(0,0,0,2)\,.\tag{80}$$

In fact the $O(4,\mathbb{C})$ algebra can now be seen to be nothing but the sum of four mutually-commuting copies of the $SL(2,\mathbb{R})$ algebra, generated by (E_i^+,E_i^-,h_i) for each $1\le i\le 4$.

B Four-charge rotating NUT solution

In this Appendix, we present the general 4-charge solution obtained by applying the $O(1,1)^4$ transformation to the Plebanski metric (56). This generalises the pairwise-equal case presented in section 5.

Making the definitions

$$r_0 = r - 2m\,, \qquad u_0 = r - 2\ell\,, \qquad \bar\rho = r\,r_0 + u\,u_0\,, \tag{81}$$

and

$$\begin{aligned}
W^2 &= r_1\,r_2\,r_3\,r_4 + u_1\,u_2\,u_3\,u_4 + 2\,u^2\,r^2 + 2\,r\,u\,(\ell\,r + m\,u)(s_1^2 + s_2^2 + s_3^2 + s_4^2)\\
&- 4\,(\ell\,r - m\,u)^2(s_{123}^2 + s_{124}^2 + s_{134}^2 + s_{234}^2 + 2\,s_{1234}^2 - 2\,c_{1234}\,s_{1234})\\
&+ 8\,m\,\ell\,r\,u\,(s_{12}^2 + s_{13}^2 + s_{14}^2 + s_{23}^2 + s_{24}^2 + s_{34}^2)\\
&+ 8\,m\,\ell(m\,u + \ell\,r)\,(s_{123}^2 + s_{124}^2 + s_{134}^2 + s_{234}^2) + 32\,m^2\,\ell^2\,s_{1234}^2
\end{aligned} \tag{82}$$

we find that in three dimensions the 4-charge solution is given by

$$\begin{aligned}
\chi_1 &= \frac{2\,(c_{24}\,s_{13} - c_{13}\,s_{24})\,(\ell\,r - m\,u)}{r_1\,r_3 + u_1\,u_3}, \quad \chi_2 = \frac{2\,(c_{14}\,s_{23} - c_{23}\,s_{14})\,(\ell\,r - m\,u)}{r_2\,r_3 + u_2\,u_3},\\
\chi_3 &= \frac{2\,(c_{34}\,s_{12} - c_{12}\,s_{34})\,(\ell\,r - m\,u)}{r_1\,r_2 + u_1\,u_2}, \quad e^{\varphi_1} = \frac{r_1\,r_3 + u_1\,u_3}{W},\\
e^{\varphi_2} &= \frac{r_2\,r_3 + u_2\,u_3}{W}, \quad e^{\varphi_3} = \frac{r_1\,r_2 + u_1\,u_2}{W}, \quad e^{\varphi_4} = \frac{r\,r_0 + u\,u_0}{W},\\
\sigma_1 &= \frac{2}{W^2}\,(\ell\,r - m\,u)\,[c_1\,s_{234}\,(r_0\,r_1 + u_0\,u_1) - s_1\,c_{234}\,(r\,r_1 + u\,u_1)],\\
\sigma_3 &= \frac{2}{W^2}\,(\ell\,r - m\,u)\,[c_3\,s_{124}\,(r_0\,r_3 + u_0\,u_3) - s_3\,c_{124}\,(r\,r_3 + u\,u_3)],\\
\sigma_2 &= \frac{2}{W^2}\Big[c_2\,s_2\Big(m\,r_1\,r_3\,r_4 + \ell\,u_1\,u_3\,u_4 + r\,u\,(\ell\,r + m\,u) + 4\,\ell\,m\,r\,u\,(s_1^2 + s_3^2 + s_4^2)\\
&\quad + 4\,\ell\,m\,(\ell\,r + m\,u)(s_{13}^2 + s_{14}^2 + s_{34}^2) + 16\,\ell^2\,m^2\,s_{134}^2\Big)\\
&\quad + 2\,(\ell\,r - m\,u)^2\Big(c_{134}\,s_{134}\,(c_2^2 + s_2^2) - c_2\,s_2\,(s_{13}^2 + s_{14}^2 + s_{34}^2 + 2\,s_{134}^2)\Big)\Big]\\
\sigma_4 &= \frac{2}{W^2}\Big[c_4\,s_4\Big(m\,r_1\,r_2\,r_3 + \ell\,u_1\,u_2\,u_3 + r\,u\,(\ell\,r + m\,u) + 4\,\ell\,m\,r\,u\,(s_1^2 + s_2^2 + s_3^2)\\
&\quad + 4\,\ell\,m\,(\ell\,r + m\,u)(s_{12}^2 + s_{13}^2 + s_{23}^2) + 16\,\ell^2\,m^2\,s_{123}^2\Big)\\
&\quad + 2\,(\ell\,r - m\,u)^2\Big(c_{123}\,s_{123}\,(c_4^2 + s_4^2) - c_4\,s_4\,(s_{12}^2 + s_{13}^2 + s_{23}^2 + 2\,s_{123}^2)\Big)\Big]\\
B_{(1)} &= \frac{2}{a\,\bar\rho}\Big[c_{1234}\Big(\ell\,u\,(a^2 + r^2) + m\,r(a^2 - u^2)\Big)\\
&\quad - s_{1234}\Big(\ell\,u_0\,(a^2 + r^2) + m\,r_0\,(a^2 - u^2) + 4\,\ell\,m\,(\ell\,r - m\,u)\Big)\Big]d\phi,\\
A_{(1)1} &= \frac{2\,c_1\,s_1}{a\,\bar\rho}\,[\,a^2\,(\ell\,r - m\,u) - r\,u\,(m\,r_0 + \ell\,u_0)\,]d\phi,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{A}^1_{(1)} &= \frac{2\,c_3\,s_3}{a\,\bar\rho}\,[\,a^2\,(\ell\,r - m\,u) - r\,u\,(m\,r_0 + \ell\,u_0)\,]d\phi,\\
A_{(1)2} &= \frac{2}{a\,\bar\rho}\Big[s_2\,c_{134}\Big(r\,u\,(\ell\,r - m\,u) + a^2\,(m\,r + \ell\,u)\Big)\\
&\qquad\qquad - c_2\,s_{134}\Big(r_0\,u_0\,(\ell\,r - m\,u) + a^2\,(m\,r_0 + \ell\,u_0)\Big)\Big]d\phi,\\
\mathcal{A}^2_{(1)} &= \frac{2}{a\,\bar\rho}\Big[s_4\,c_{123}\Big(r\,u\,(\ell\,r - m\,u) + a^2\,(m\,r + \ell\,u)\Big)\\
&\qquad\qquad - c_4\,s_{123}\Big(r_0\,u_0\,(\ell\,r - m\,u) + a^2\,(m\,r_0 + \ell\,u_0)\Big)\Big]d\phi\,.
\end{aligned}
\tag{83}
$$

Lifting back to four dimensions, the metric is given by

$$
ds_4^2 = -\frac{\bar\rho}{W}\,(dt + B_{(1)})^2 + W\,\Big(\frac{dr^2}{\Delta_r} + \frac{du^2}{\Delta_u} + \frac{\Delta_r\,\Delta_u}{a^2\,\bar\rho}\,d\phi^2\Big)\,, \tag{84}
$$

and the 4 four-dimensional gauge potentials are given in terms of the three-dimensional expressions in (83) by

$$
\begin{aligned}
\hat A_{(1)1} &= (A_{(1)1} + \sigma_1\,\mathcal{B}_{(1)}) + \sigma_1\,dt\,,\\
\hat A_{(1)2} &= (A_{(1)2} + \sigma_2\,\mathcal{B}_{(1)}) + \sigma_2\,dt\,,\\
\hat{\mathcal{A}}^1_{(1)} &= (\mathcal{A}^1_{(1)} + \sigma_3\,\mathcal{B}_{(1)}) + \sigma_3\,dt\,,\\
\hat{\mathcal{A}}^2_{(1)} &= (\mathcal{A}^2_{(1)} + \sigma_4\,\mathcal{B}_{(1)}) + \sigma_4\,dt\,.
\end{aligned}
\tag{85}
$$

The dilatons $(\varphi_1, \varphi_2, \varphi_3)$ and axions (χ_1, χ_2, χ_3) are simply given by their three-dimensional expressions in (83).

C Supersymmetry of Rotating AdS Black Holes

A complete discussion of the supersymmetry of the charged solutions we have constructed is rather involved, and we shall not present it here. There should be no essential difference between the supersymmetry of the multi-charge solutions and the case where all charges are set equal. Thus for the purpose of discussions the fractions of supersymmetry that are preserved, it suffices to consider just the case of charged rotating black holes in gauged Einstein-Maxwell supergravity. The results are mostly known from previous literature, but it is perhaps useful to collect them together here.

Setting all the charges equal, in framework where the two magnetically-charge fields have been dualised to electrically-charged fields, as in (52) and (53), our solution reduces to the standard Kerr-Newman charged rotating black hole with a cosmological constant. This is given by

$$
ds_4^2 = -\frac{\Delta_r}{\rho^2}\,(dt - a\,\sin^2\theta\,d\phi)^2 + \rho^2\,\Big(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\Big) + \frac{\Delta_\theta\,\sin^2\theta}{\rho^2}\,[a\,dt - (r^2 + a^2)d\phi]^2\,,
$$

$$A_{(1)} \quad = \quad \frac{2Q\,r}{\rho^2}[dt - a\,\sin^2\theta\,d\phi]\,, \tag{86}$$

where

$$\rho^2 = r^2 + a^2\,\cos^2\theta\,, \quad \Delta_r = (r^2+a^2)(1+g^2\,r^2) - 2m\,r + Q^2\,, \quad \Delta_\theta = 1 - g^2\,a^2\,\cos^2\theta\,. \tag{87}$$

This satisfies the equations of motion following from the Lagrangian

$$\mathcal{L} = e\,(R - \tfrac{1}{4}F_{(2)}^2 - 6g^2)\,. \tag{88}$$

Equation (88) is the bosonic sector of Einstein-Maxwell supergravity, for which the gravitino transformation rule is $\delta\psi_\mu = D_\mu\,\epsilon$, where

$$D_\mu = \nabla_\mu - \tfrac{i}{2}\,g\,A_\mu + \tfrac{i}{8}\,F_{\nu\rho}\,\Gamma^{\nu\rho}\,\Gamma_\mu + \tfrac{1}{2}g\,\Gamma_\mu\,. \tag{89}$$

From this, we can easily calculate the integrability conditions $M_{\mu\nu}\,\eta \equiv [D_\mu\,D_\nu]\,\eta = 0$ for the existence of a Killing spinor η. The fraction of preserved supersymmetry is determined by the number of common zero-eigenvalue eigenspinors of the set of matrices $M_{\mu\nu}$. We find the following:

$\underline{g = 0,\ a = 0}$:

In this ungauged case, with no rotation, there are two common zero eigenvalues if $Q = m$, and two if $Q = -m$. Thus a fraction $\frac{1}{2}$ of the supersymmetry is preserved.

$\underline{g = 0,\ a \neq 0}$:

In this ungauged case with rotation, there are again two common zero eigenvalues if $Q = m$ or $Q = -m$, and so again $\frac{1}{2}$ supersymmetry is preserved.

$\underline{g \neq 0,\ a = 0}$:

In this gauged case without rotation, there are again two common zero eigenvalues if $Q = m$ or $Q = -m$, and again $\frac{1}{2}$ supersymmetry is preserved.

$\underline{g \neq 0,\ a \neq 0}$:

In the gauged case with rotation, there is just one common zero eigenvalue, arising in any of the four cases $m = \pm Q \pm a\,g\,Q$. Thus four any of these four sign choices, we find $\frac{1}{4}$ of the supersymmetry is preserved.

It is worth remarking that our result in the case of $g \neq 0$ and $a \neq 0$ (which also encompasses the previous three specialisations) is in complete agreement with the discussion in Kostelecky and Perry [26].

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